A TRANSFORMATIVE LEARNING EXPERIENCE FOR THE CONCEPT OF FUNCTION

Michael S Meagher
Brooklyn College - CUNY
mmeagher@brooklyn.cuny.edu

Allison McCulloch
UNC Charlotte
amccul11@uncc.edu

Jennifer Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Milan F Sherman
Drake University
milan.sherman@drake.edu

Functions form a central part of the U.S. mathematics curriculum. A large body of research shows that students at all levels, including preservice secondary mathematics teachers, have difficulties with defining function as a correspondence between two sets with a univalence condition. Those difficulties include privileging algebraic representations and reductive interpretations of the univalence condition in the form of the vertical line test. In our research study, 47 pre-service mathematics teachers provided definitions of function, engaged with an interactive applet that had a non-standard representation of function. ***The interaction with the applet was effective in initiating a series of dilemmas in their conception of function that resulted in the majority of the participants transforming and/or refining their conception of function.***

Keywords: Teacher Education - Preservice, Teacher Knowledge, Technology

Introduction

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Students work with functions from the very earliest grades in pattern exploration, through high school with a formal treatment of functions as arbitrary mappings between sets. There is an extensive body of research on students' understanding of function (e.g., Carlson et al., 2003; Cooney et al., 2010; Dubinsky & Harel, 1992; Even, 1990; 1993, Oehrtman et al., 2008) and much of that research reports that learners (including preservice mathematics teachers) have considerable difficulty identifying functions and in distinguishing them from non-functions. With the emphasis on function in school mathematics, preservice secondary mathematics teachers (PSMTs) must have robust conceptions of function to plan to support their students' understanding. Based on years of research on students and teachers flawed, limited conceptions of function, and on transformation theory (Mezirow, 2000; Taylor, 2007), we designed a task, utilizing advanced digital technology, to meet this need. The purpose of this study is to examine ways in which the task elicited and transformed PSMTs' conceptions of function.

Literature Review and Relationship to Research

Defining Function

In Thompson & Carlson's (2017) discussion of the evolution of the definition of function, they describe how a variation and covariation conception of function came to be replaced, owing to the emerging dominance of a set theoretic conception of variable as used in group theory and other areas, by a correspondence conception of function. This definition "solved problems that arose for mathematicians, [but that] introducing it in school mathematics made it nearly

impossible for school students to see any intellectual need for it" (p.422). This abstract correspondence definition is often referred to as the Dirichlet-Bourbaki definition of function and states that a function is a correspondence between arbitrary sets satisfying a univalence condition i.e. each element in the domain corresponds to exactly one element in the codomain.

Thompson and Carlson (2017), citing Cooney and Wilson (1993), as well as drawing on their own review of 17 U.S. Precalculus textbooks, note that a correspondence definition of function is used exclusively in all of these textbooks. Therefore, while we expect that most students (and PSMTs) who have attended U.S. schools to have experience with a definition involving a correspondence between two sets with constraints on the mapping of individual elements (the univalence condition), Even (1993) notes that many students retain a "protypic" (p.96) concept of functions as linear relationships and "many expect graphs of functions to be "reasonable" and functions to be representable by a formula." (p. 96).

Teachers' Understandings of the Function Concept

In addition to content knowledge of functions, mathematics teachers require Mathematical Knowledge for Teaching (Ball, Hill, & Bass, 2005) of functions i.e. teachers should be aware of various representations of functions, many examples of functions and non-functions, and known areas of challenge for students when learning functions. However, teachers' understanding of function, at the content level, is similar to that of school and college students (Bannister, 2014; Even, 1990, 1993; Wilson, 1994). In particular, practicing teachers and PSMTs tend to privilege algebraic representations of functions and emphasize properties of graphs (e.g., vertical line test) in their descriptions of functions and non-functions (Even, 1990, 1993; Wilson, 1994). They also exhibit a limited repertoire of representations on which to draw in helping students understand functions (Bannister, 2014; Hatisaru & Erbas, 2017).

Crucially, teachers' understanding of function has been shown to impact the pedagogical choices they make during instruction. In a study of 152 PSMTs, Even (1993) found they could not justify the need for univalence and did not know why it was important to distinguish between functions and non-functions. Owing to this lack of content knowledge, the PSMTs' MKT was constrained and they limited the exposure of their students to various function representations and emphasized procedures such as the vertical line test in identifying functions.

Learning in Technology-rich Environments

A considerable body of research supports the idea that advanced digital technologies can support learning in general (Tamim et al., 2011) and mathematics concepts in particular (Drijvers et al., 2010; Olive et al., 2010.). Drijvers (2015) argues that "the criterion for appropriate design is that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake." (p.15). For example, a good design to allow students to engage with the concept of function will allow the user to experience different kinds of functions and non-functions with enough data to differentiate between the two.

Theoretical Framework

Transformation Theory

Given the preponderance of evidence in the literature that the conception of function of PSMTs is often underdeveloped, our goal was to design a learning experience that problematized those conceptions, required PSMTs to reflect on them, and, ideally, resulted in refinement of their conception. Given that PSMTs come to their methods courses as adults and with a wealth of

previous experiences related to the function concept, we turned to an adult learning theory, namely Mezirow's (2000) transformation theory. Transformation theory is an adult learning theory that is consistent with constructivist assumptions and expands on those assumptions by acknowledging the broad predispositions an adult might have toward a concept based on prior experiences, and the role these dispositions play in their meaning making (Mezirow, 2000).

Mezirow (2009) describes four forms of learning at the heart of transformation theory: elaborating existing meaning schemes, creating new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. According to Mezirow (2009), learning by transforming existing meaning schemes and perspectives often begins with a stimulus, a *disorienting dilemma*, which requires one to question their current meaning schemes. However, experiencing a disorienting dilemma alone is not enough to effect a transformation and learning will only occur through critical reflection (Merriam, 2004; Mezirow, 2000; Taylor, 2007).

Sfard (1991) distinguishes between a *concept,* a mathematical idea in its "official form" (p. 3) and a *conception*, "the whole cluster internal representations and associations evoked by a concept" (p. 3). A *concept* is the generally accepted structure of mathematics that has been culturally developed and shared formally among mathematicians for centuries (Pehkonen & Pietila, 2003) whereas a *conception* is a learner's individual, often, incomplete understanding of the concept. In Sfard's definition of conception we see the connection between knowledge and the affective aspects of meaning schemes as defined by Mezirow (2000). Conceptions, then, are the personal side of a concept, one's individual experiences, beliefs, attitudes, and emotions that result in personal definitions, examples, and non-examples of concepts.

Therefore, we aimed to design a learning experience for PSMTs that would trigger a disorienting dilemma related to their conceptions of function and non-function and require critical reflection in the expectation that a transformation of meaning scheme occurs.

A Transformative Learning Experience for the Concept of Function

Given the promise of cognitive roots (Tall et al., 2000) such as a function machine, we set out to design a machine-based experience using representations that were unfamiliar for PSMTs as a stimulus for examining their meaning schemes of function. The applet we designed, built on the metaphor of a vending machine, contained no numerical or algebraic expressions. Our intention was to put PSMTs in a context in which they would not be able to automatically rely on an algebraic, and often procedural, conceptions of functions (e.g., use of the vertical line test).

The Vending Machine applet (https://ggbm.at/X3Cn7npQ) consists of four pages; each with two to six vending machines and asks the user to identify each vending machine as a function or non-function. The machines each consist of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is pressed it produces none, one, or more than one of the different colored cans which may, or may not, correspond to the color of the button pressed.

Which One is a Function? (pgs. 1 - 3)			
A	Red → Red Blue → Blue Silver → Silver Green → Green	B	Red → Red Blue → Blue Silver → Silver Green → Random
C	Red → Blue Blue → Silver Silver → Green Green → Red	D	Red → Random Pair Blue → Blue Silver → Silver Green → Green
E	Red → Red Blue → Blue & Random Silver → Silver Green → Green	F	Red → Red Blue → Silver Silver → Silver Green → Green

Which Ones Could be Functions? (pg. 4)			
G	All Random	H	Red → Green Blue → Green Silver → Green Green → Green
I	Red → 2 Silvers Blue → Green Silver → Red Green → Blue	J	Red → Red Blue → Blue Silver → No Can Green → Green
K	Red → Red Blue → Blue Silver → Silver Green → Red & Green	L	Red → Red Blue → Red Silver → Silver Green → Silver

Figure 1: Description of Each Machine in the Applet

By removing numeric and algebraic representations, the applet could allow PSMTs to attend to input/outputs and their relationship. We intentionally designed to trigger dilemmas related to known issues from the literature e.g. researchers have shown that students as well as teachers exhibit difficulties identifying constant functions as functions (e.g., Carlson, 1998; Rasmussen, 2000); thus, there is a machine that acts as a constant function, i.e. every button produces the same color can. The purpose of this study is to determine the extent to which we were successful in designing for transformative learning (Mezirow, 2000; Taylor, 2007) related to the function definition. Specifically, we aim to answer the following research question:

> To what extent did PSMTs experience disorienting dilemmas when engaging with the vending machine applet and in what ways did PSMTs' conceptions of function transform as a result of personal critical reflection in the context of the vending machine task?

Methods

Participants and Data Sources

Participants in this study are 47 PSMTs enrolled in a secondary mathematics methods course at four different U.S. universities, ranging from five to 18 PSMTs per university. The PSMTs were all undergraduate mathematics and/or mathematics education majors working toward a secondary mathematics teaching license. The individual degree programs all required at least 36 hours of mathematics, and the PSMTs had all completed at least Calculus II at the time of the study. Every PSMT in the four methods courses took part in the study (N = 55). However, there were some PSMTs that did not have complete data sets (e.g., video had no sound, missing artifacts), these participants were removed, leaving 47 PSMTs in this particular study.

Data for this study consists of all of the PSMTs' work related to the Vending Machine task. During a class the PSMTs completed the task individually and then the class came together for a whole class discussion. Data is focused on PSMTs individual work on the task and includes both written work and video recorded screen captures of their engagement with the applet. Specifically, we collected PSMTs' written pre- and post-definitions of function and their written responses to the Vending Machine task worksheet. In addition, each PSMT captured a screencast of their work on the Vending Machine task as they followed a "think-aloud" protocol while working on the task.

Analysis of Applet Engagement Descriptions

We created a document for each PSMT which consisted of their pre- and post-definitions and a detailed description of the video-recorded screencast. These descriptions included a

chronological record of PSMTs' engagement with the applet and verbatim transcriptions of PSMTs' expressed thoughts related to their work. The participant descriptions were uploaded to Atlas.ti and coded for evidence of the occurrence of disorienting dilemmas, triggers for those dilemmas, articulated conceptions of function that were challenged by those triggers, and transformations of meaning schemes related to conceptions.

Dilemmas were identified based on PSMTs' verbal utterances and interactions with the applet. For example, verbal utterances such as, "Ok these are two cans but they seem to be the same. Does it have to be one can of coke, or two cans can still be one output? I don't know. Let's see what others are" were coded as a dilemma. Each dilemma was then assigned a trigger code. Each dilemma was either resolved or not resolved in the PSMTs' utterances. Those that were resolved were coded as having a change in meaning scheme. Trigger codes included both a priori triggers and emergent triggers. Given the personal nature of conceptions of function, we could only code for those that were articulated explicitly in writing or spoken on the screencast. All of the quotations coded for function conception were then open coded to identify themes (Creswell, 2014). Themes for the articulated conceptions of function are shown were: (i) Functions must be/don't have to be continuous; (ii) Functions can/can't be many-to-one; (iii) Functions must map elements in the domain to elements in a defined codomain; and (iv) Functions must be one to one (with or without correct meaning of one to one).

Findings

In our findings we discuss the dilemmas triggered through engagement with the applet, the conceptions of function which were problematized, and the ways in which PSMTs' conceptions of function were transformed as a result of engaging with the Vending Machine Task.

Triggering Disorienting Dilemmas and Transforming Conceptions

From our analysis of the 47 PSMTs' screencast descriptions, we identified a total of 158 dilemmas (i.e. a little over 3 per PSMT on average), with approximately 91% (43 PSMTs) articulating at least one dilemma while engaging with the applet. The machines that triggered these dilemmas were those that produced two cans or no can as an output, those for which different inputs produced the same output, or those for which the color of the can (output) did not match the button pressed (input). Of the 158 articulated dilemmas, 124 resulted in a transformation in a PSMTs' conception of function, where transformation is defined, per our theoretical framework as any one of: elaborating existing meaning schemes, creating new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. The transformations we report on below, i.e. the resolved dilemmas, were mainly "elaborating existing meaning schemes" although there were some instances of "transforming meaning schemes." The three most commonly transformed conception themes were: elements of the codomain, many-to-one, and continuous function. For the sake of brevity, only the first of these will be discussed in this paper.

Conceptions of Elements in the Codomain

The most common conception of function that was triggered was that of PSMTs considering (or not) possible ways of defining the codomain, with 41 of the 47 PSMTs (87%) articulating such a dilemma. We intentionally did not state what elements made up the domain, codomain, and range for the machines in the applet. The machines that produced two cans as an output (D, E, I, K) and no can as an output (J) were designed with the hope that engagement with the machines would trigger a dilemma for the PSMTs and elicit such considerations.

When encountering Machines D, E, I, K, and J, PSMTs typically articulated their dilemma by questioning whether or not no can or two cans could be outputs. For example, PSMT 2's response to Silver Mist not producing an output on Machine J, "Oh, the Silver Mist has no output. [Presses Silver Mist button many times.] That's not broken, right? So, is that ok, that Silver Mist doesn't have an output?" and PSMT 6's response to two cans as an output on Machine K, "Red is red, blue is blue, silver is silver, and green is [red and green cans appear]. That is definitely not a function, because you can't have two. I guess you could have two outputs (sigh) this is very difficult." Additionally, some PSMTs went beyond just verbalizing a dilemma to explicitly discussing the possible output values. One example that typifies how PSMTs explicitly discussed defining the possible codomain elements follows,

> Ok these are two cans, but they seem to be the same. That's an interesting question… That's interesting what we define as output. Does it have to be one can of coke or two cans can still be one output? (PSMT 3, on Machine I).

All of these PSMTs clearly articulated a dilemma related to their conception of types of elements that could possibly be in the codomain.

The first machine where these dilemmas occurred was on Machine D (Red Cola → random pair), and while 20 PSMTs articulated a dilemma on this machine, only 6 resulted in a transformed conception of function. This result can be attributed to the fact that making sense of the two can output was not required to classify the machine as a function or non-function due to the randomness of the output. For example, after deciding that the machine was not a function because of the random nature of the outputs PSMT 23 stated, "And, I'm still hung up on this two can thing, but I really don't know why. And I haven't been able to work through it yet. So, maybe I can explore some more and get back to that."

Most transformations of conceptions of functions related to defining elements of the codomain occurred on Machines I, J, and K (17, 20, and 18 respectively). As PSMTs articulated their transforming conceptions related to the nature of the codomain they either focused on the consistency of the outputs, examples of representations of functions they were familiar with, or their personal definition of function.

Consistency of outputs. Most PSMTs (23 out of 41) who articulated transformed conceptions with respect to codomain attended to consistency as they worked to make sense of what they observed as outputs on Machines I, J, and K. For example, as PSMT 7 engaged with Machine I they explained,

> Ok, now I'm having second thoughts about these two sodas. And like, would Green Dew have two different arrows? So, I guess it depends on how you see your output values. Are the output values just a red soda, green soda, blue soda, silver soda? Or can they be different combinations of those? So, going off of the assumption that it's going off the same output every time, then it's a function. But since it's giving you two different drinks, is it still? Hmm, I'm questioning all my thoughts now. I guess it would depend on how you classify your outputs, so if like getting two different drinks is OK, but as long as it happens every single time that you put this input in then I think it would be OK.

In this example PSMT 7 is considering elements that might be in the codomain, going beyond noting elements of the range that have been observed with the machines so far. PSMT 7 makes

the point that whether or not this is a function, depends on how the codomain is defined, ultimately deciding that if it was defined to include pairs of cans then Machine I could be a function because the result of clicking on the Red Cola button is consistently two silver cans. Similar reasoning is evident in the following explanation of Machine J by PSMT 30:

> That's not broken, right? So, is that ok, that silver mist doesn't have an output? … I think its ok, because it's the same output. If it gave us something one time, then I wouldn't be ok with that. So, even though silver doesn't give you anything, by giving you nothing, it is consistently giving you nothing.

In each of these responses, the PSMTs articulated transformed conceptions of the codomain, specifically an elaborated conception that included two cans or no cans as elements. This transformation is a result of considering the importance of consistency in the relationship between input and output elements. Furthermore, all of the PSMTs that attended to consistency when deliberating about how to resolve their dilemma transformed their conception of codomain to include both two and no cans.

Using examples of familiar functions. Other PSMTs worked through dilemmas in which their conceptions of the nature of codomain were challenged by drawing upon examples of familiar representations of function. This is evident in PSMT 34's work on Machine I,

> This one is the most questionable one that I'm the least certain on. We will say that is not a function. I don't know how that would really work on a graph. How that could be expressed as a graph? I think that would basically be saying if I put in a one I would get two 2s out of that. Which is not possible.

PSMT 34 could not imagine how two cans might be represented on a graph, resulting in a transformed conception of codomain in this context that did not include elements other than single cans. PSMT 11 used similar reasoning on Machine J (no can),

> Silver doesn't give me anything. What? … I don't know, it's just the fact that it doesn't give me one out, if that's the reason why I don't think it's a function or…even like a linear basic function. If you put something into it, you have to come out with something. There's got to be some number there. Okay, okay, because you have to have an output.

In this example, the PSMT is trying to imagine the situation as a known function, even a "linear basic function" but is not able to do so. After declaring they cannot think of a function that behaves this way, the PSMT goes further to state that an input must have an output to go with it.

When PSMTs drew on familiar representations to make sense of a dilemma regarding issues with the codomain, they used the familiar representations to determine if a machine was a function or non-function. This resulted in a changed understanding of the codomain in this non-algebraic context by drawing on possible codomains from algebraic contexts.

Drawing on personal definition of function. Drawing upon one's personal definition was also common for the PSMTs when they were thinking about elements of the range and codomain. Consider PSMT 20's explanation of Machine I,

> Immediately that red button is giving you two different cans. Which is not… They are both silvers, but I take that as still two different values even if they're the same value which you can't have. Yeah two different cans we can't have two cans off of one… two ys off one x. Then it's not a function. Although if they are the same can… nah I still think that's not.

Similarly, PSMT 35 working on Machine J said,

> The problem is the silver, because it since it doesn't give you anything. It's like having an x that doesn't go to a y. And, one of the rules of functions is that every x needs a y but not every y has to have an x. So, because silver doesn't give you a can, this makes it not a function.

Both of these examples are evidence of PSMTs transforming their conceptions of codomain. In the latter case, in such a way that the empty set is not included based on their conceptions of the univalence requirement of the definition of function.

Discussion

The purpose of the design of the Vending Machine applet and this study was to elicit PSMTs' many conceptions of functions and attempt to challenge and transform any less robust conceptions to more robust conceptions of functions. The fact that PSMTs exhibited many conceptions and difficulties consistent with research literature on understanding of function suggest that the applet design was effective in this regard. Furthermore, there is evidence in the totality of the data collected that engaging with the Vending Machine applet resulted in most PSMTs reconsidering and refining their conceptions of function in a positive direction.

To enable and facilitate this change we drew on transformation theory (Mezirow, 2000), and Drijvers's (2015) notion of the importance of didactical possibilities in the design of advanced digital technology applications, to guide the creation of an applet to trigger dilemmas that address common conceptions from the literature on distinguishing functions and non-functions. The use of advanced digital technology, allowed us to create a task with which the PSMTs could interact independently and which, with immediate feedback allowed them to formulate conjectures and test those conjectures without having to wait for a class discussion or intervention from an instructor. At the beginning of the task PSMTs articulated conceptions of function that we expected based on the literature. For example, the PSMTs articulated disorienting dilemmas related to their conceptions of univalence (e.g., Even, 1993; Vinner & Dreyfus, 1989), many to one (e.g., Carlson, 1998; Rasmussen, 2000), and continuity (Bezuidenhout, 2001; Tall & Vinner, 1981). With the exception of continuity, we designed the applet to trigger dilemmas related to each of these conceptions. One of the persistent problems noted in the literature is over privileging of algebraic representations (Even, 1990, 1993; Wilson, 1994), putting PSMTs in the context of the vending machines appears to have mitigated this problem.

The ways in which the PSMTs engaged with the applet provided considerable insight into their conceptions of functions and the ways in which they were both challenged and transformed. Given that 43 out of 47 (91%) PSMTs articulated a dilemma related to at least one of the triggers we designed for, we know that we leveraged advanced digital technology to create an opportunity for transformations of conceptions to occur. Furthermore, 41 of the 47 PSMTs (87%) experienced a dilemma related to their conceptions of codomain and the result that 76%

(31 out of those 41) were able to articulate the impact of the definition of domain and codomain on the way the machines would be classified, is significant. Overall, the vending machine task was successful in triggering dilemmas, eliciting critical reflection, and supporting PSMTs in transforming their conceptions of function.

Conclusion

It is crucial that PSMTs have a solid understanding of function, know variations in the definition of function, develop the ability to translate among different representations of functions, and know when to use each definition based on context (Bannister, 2014; Hatisaru & Erbas, 2017). This specialized content knowledge is needed to understand and plan for the diverse student conceptions they will encounter during instruction. While there is a vast literature base on the limited conceptions of functions PSMTs often develop through high school and undergraduate mathematics, little is known about how to transform them after years of building on them in algebraic contexts. The results of this study indicate that by removing PSMTs from familiar function contexts and designing to trigger dilemmas based on conceptions identified in the literature, we can transform PSMTs' conceptions of function in a positive direction.

Acknowledgments

This work was partially supported by the National Science Foundation (NSF) under grant DUE 1820998 awarded to Middle Tennessee State University, grant DUE 1821054 awarded to University of North Carolina at Charlotte, and grant DUE 11230001 awarded to NC State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the NSF.

References

Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(3), 14-22.

Bannister, V. R. P. (2014). Flexible conceptions of perspectives and representations: An examination of pre-service mathematics teachers' knowledge. *International Journal of Education in Mathematics, Science and Technology*, *2*(3), 223-233.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III, Issues in mathematics education, 7*(1), 115-162. American Mathematical Society.

Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PME-NA* (Vol 2, pp. 165-172). Honolulu, HI: University of Hawaii.

Cooney, T. J., Beckman, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 131–158). Hillsdale, NJ: Erlbaum.

Creswell, J. (2014). *Research design: qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks, California: SAGE Publications.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In J. S. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 135-151). Cham: Springer.

Dubinsky, E., & Harel, G. (1992). *The concept of function: Aspects of epistemology and pedagogy*. Washington, D.C.: Mathematical Association of America.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521-544.
Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94-116.
Hatisaru, V., & Erbas, A. K. (2017). Mathematical knowledge for teaching the function concept and student learning outcomes. *International Journal of Science and Mathematics Education, 15*, 703-722. doi:10.1007/s10763-015-9707-5
Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.
Merriam, S. B. (2004). The role of cognitive development in Mezirow's transformational learning theory. *Adult Education Quarterly*, *55*(1), 60-68.
Mezirow, J. (2000). Learning to think like an adult: Core concepts of transformation theory. In J. Merizow & Associates (Ed.), *Learning as transformation: Critical perspectives on a theory in progress* (pp. 3-34). San Francisco, CA: Jossey-Bass.
Mezirow, J. (2009). Transformative learning theory. In J. Merizow & E. W. Taylor (Eds.), *Transformative learning in practice: Insights from community, workplace, and higher education* (pp. 18-31). San Francisco: Jossey-Bass.
Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.
Olive, J., Makar, K., Hoyos, V., Kor, L. K., Kosheleva, O., & Sträßer, R. (2010). Mathematical knowledge and practices resulting from access to digital technologies. *Mathematics education and technology: Rethinking the terrain: the 17th ICMI Study* (Vol. 13, pp. 133-177) New York, NY: Springer.
Pehkonen, E., & Pietilä, A. (2004). On relationships between beliefs and knowledge in mathematics education. In M. Mariotti (Ed.), *Proceedings of the Third Congress of European Society for Research in Mathematics Education (CD/ROM)*. Italy: University of Pisa. Retrieved from: http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG2/TG2_pehkonen_cerme3.pdf
Rasmussen, C. L. (2000). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20,* 55-87.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on the processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.
Tall, D., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for the function concept. In M. L. Fernandez (Ed.), *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 255-261).
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.
Tamim, R. M., Bernard, R. M., Borokhovski, E., Abrami, P. C., & Schmid, R. F. (2011). What 40 years of research says about the impact of technology on learning: A second-order meta-analysis and validation study. *Review of Educational Research, 81*(1), 4–28.
Taylor, E. W. (2007). An update of transformative learning theory: A critical review of the empirical research (1999-2005). *International Journal of Lifelong Education, 26*, 173-191.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356-366.
Wilson, M. R. (1994). One preservice secondary teachers' understanding of function: The impact of course integrating mathematical content and pedagogy. *Journal for Research in Mathematics E*